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COMMENT

A direct proof of Kim's identities

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Abstract. As a by-product of a finite-size Bethe ansatz calculation in statistical mechanics, Kim has established, by an indirect route, three mathematical identities rather similar to the conjugate modulus relations satisfied by the elliptic theta constants. However, they contain factors such as $1 - q^{\sqrt{n}}$ and $1 - q^{n^2}$, instead of $1 - q^n$. We show that there is a fourth relation that naturally completes the set, in much the same way as there are four relations for the four elliptic theta functions. We derive all of them directly by proving and using a specialization of Weierstrass' factorization theorem in complex variable theory.

1. Introduction

Kim (1996) has obtained the leading finite-size corrections to the spectra of the asymmetric XXZ chain and the related six-vertex model, near the antiferromagnetic phase boundary at zero vertical field. He also performed the calculation at zero horizontal field. These results are related by a 90° rotation, which yields the following three identities, true for all real positive τ:

$$\prod_{n=-\infty}^{\infty} [1 + p^{(2n-1)^2}] = ((q\bar{q})^{-c} \prod_{n=1}^{\infty} (1 + q^{\sqrt{2n-1}})^2 (1 + \bar{q}^{\sqrt{2n-1}})^2 \tag{1}$$

$$\prod_{n=-\infty}^{\infty} (1 + p^{4n^2}) = (q\bar{q})^{-c} \prod_{n=1}^{\infty} (1 - q^{\sqrt{2n-1}})^2 (1 - \bar{q}^{\sqrt{2n-1}})^2 \tag{2}$$

$$\prod_{n=-\infty}^{\infty} [1 - p^{(2n-1)^2}] = 4(q\bar{q})^{d} \prod_{n=1}^{\infty} (1 + q^{\sqrt{2n}})^2 \ (1 + \bar{q}^{\sqrt{2n}})^2. \tag{3}$$

Here

$$c = (\sqrt{2} - 1)\zeta(\tfrac{3}{2})/(4\pi) = 0.086\,109\,29\ldots$$

$$d = \zeta(\tfrac{3}{2})/(2\pi\sqrt{2}) = 0.293\,995\,52\ldots \qquad p = \mathrm{e}^{-\pi/\tau}$$

$$q = \mathrm{e}^{-\pi\sqrt{\mathrm{i}\tau}} \qquad \bar{q} = \mathrm{e}^{-\pi\sqrt{-\mathrm{i}\tau}}.$$

All square roots herein are chosen to be in the right half plane. By q^λ, $\bar{q}^\lambda$ we mean $\mathrm{e}^{-\pi\lambda\sqrt{\mathrm{i}\tau}}$, $\mathrm{e}^{-\pi\lambda\sqrt{-\mathrm{i}\tau}}$, respectively.

These identities (1)–(3) are reminiscent of the conjugate modulus identities of the elliptic theta constants $H_1(0), \Theta(0), \Theta_1(0)$ (section 21.51 of Whittaker and Watson 1950, p 75 of Courant and Hilbert 1953, equation 15.7.2 of Baxter 1982). However, in those identities q (or rather $\mathrm{e}^{-\pi\tau}$) and p are raised to a power proportional to n, rather than n^2 or $\sqrt{n}$.

We would like to have a direct proof of (1)–(3). We obtain one here by using the Poisson transformation and complex variable theory to establish the general result (5). We specialize this to (14), from which we obtain a fourth identity:

$$\prod_{n=1}^{\infty}(1-p^{4n^2})^2 = \pi\tau(q\overline{q})^d \prod_{n=1}^{\infty}(1-q^{\sqrt{2n}})^2(1-\overline{q}^{\sqrt{2n}})^2. \tag{4}$$

This is analogous to the conjugate modulus identity for $H'(0)$.

We can write the products in (1)–(3) in terms of the type of products occurring in (4) by using the elementary identities $1+x = (1-x^2)/(1-x)$ and $\prod_n f(2n-1) = \prod_n [f(n)/f(2n)]$. Write identity (j), for a given value of τ, as (j,τ). Then in this way we find that $(2,\tau)$ can be obtained from the ratio $(4,\tau/2) : (4,\tau)$. Also, $(3,\tau)$ follows from $(4,4\tau)$:$(4,\tau)$. Finally, $(1,\tau)$ can be obtained from the ratio $(2,4\tau)$:$(2,\tau)$, or alternatively from $(3,\tau/2)$:$(3,\tau)$. Thus (4) implies (1)–(3).

2. Proof of identity (4)

We begin by proving a general theorem. Let $F(z)$ be a meromorphic function of a complex variable such that:

(i) $\log F(z)$ is analytic on the real axis and $\log F(x)$ is Fourier analysable;

(ii) the integral $\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} F'(x)/F(x)$ can be closed round the upper half plane for $\mathrm{Re}\,k > 0$, round the lower half plane for $\mathrm{Re}\,k < 0$ (i.e. there exists a discrete sequence of ever increasing appropriate arcs such that the integral over the arc tends to zero);

(iii) the zeros and poles of $F(z)$ in the UHP are at $u_1, u_2, \ldots$, and m_r is the multiplicity of the zero at u_r (regarding poles as zeros with negative multiplicity: thus a pole of order j has multiplicity $-j$). Similarly, the zeros and poles in the LHP are at $v_1, v_2, \ldots$, with multiplicities $n_1, n_2, \ldots$.

Then, for all real positive x and δ,

$$\prod_{n=-\infty}^{\infty} F(x+n\delta) = \exp\left[\delta^{-1}\int_{-\infty}^{\infty} \log F(t)\,\mathrm{d}t\right] \prod_r [1-\mathrm{e}^{2\pi\mathrm{i}(u_r-x)/\delta}]^{m_r} \prod_r [1-\mathrm{e}^{2\pi\mathrm{i}(x-v_r)/\delta}]^{n_r} \tag{5}$$

the first product on the RHS being over the zeros and poles in the UHP, the second over those in the LHP.

This identity is a variant of Weierstrass' factorization theorem (section 7.6 of Whittaker and Watson 1950). One can readily verify that both sides, considered as functions of x, have the same zeros and poles.

Proof. To prove this theorem, let $g(k)$ be the Fourier transform of $\log F(x)$:

$$g(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \log F(x)\,\mathrm{d}x. \tag{6}$$

By 'Fourier analysable' in requirement (i) we mean that this integral is absolutely convergent, for all real k.

Then from the Poisson transform (pp 75–7 of Courant and Hilbert 1953, equation 15.8.1 of Baxter 1982) applied to the function $\log F(x+a)$, where a is an arbitrary real parameter,

$$\sum_{n=-\infty}^{\infty} \log F(a+n\delta) = \delta^{-1} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}na/\delta} g(2\pi n/\delta). \tag{7}$$

Integrating (6) by parts, noting that $\log F(x)$ necessarily tends to zero as $x \to \pm\infty$,

$$g(k) = (i/k) \int_{-\infty}^{\infty} e^{ikx} F'(x)/F(x)\, dx. \tag{8}$$

If $k > 0$, it follows from requirement (ii) that

$$g(k) = -\frac{2\pi}{k} \sum_r m_r e^{iku_r} \tag{9}$$

while for $k < 0$

$$g(k) = \frac{2\pi}{k} \sum_r n_r e^{ikv_r} \tag{10}$$

the sums being over the zeros u_r and v_r, respectively.

Substituting these expressions into the RHS of (7) and interchanging the order of the summations, we obtain

$$\sum_{n=-\infty}^{\infty} \log F(a + n\delta) = \frac{g(0)}{\delta} + \sum_r m_r \log[1 - e^{2\pi i(u_r - a)/\delta}] + \sum_r n_r \log[1 - e^{2\pi i(a - v_r)/\delta}]. \tag{11}$$

Exponentiating and replacing a by x, we obtain the desired result (5). □

Corollaries. Let y be real, in the range $-1 < y < 1$. Define

$$s = e^{i\pi y} \qquad w = e^{i\pi x} \tag{12}$$

$$F(z) = 1 + s e^{-\pi z^2/\tau}. \tag{13}$$

Regard y and τ as constants: then $F(z)$ is entire, with only simple zeros, and satisfies conditions (i)–(iii). Set $\delta = 2$ and use the above definitions of p, q, $\overline{q}$. Then (5) becomes, for all real x,

$$\prod_{n=-\infty}^{\infty} [1 + sp^{(2n+x)^2}] = (q\overline{q})^{-h(s)} \prod_{n=1}^{\infty} \{[1 - wq^{\sqrt{2n-1-y}}][1 - w^{-1}q^{\sqrt{2n-1-y}}]$$
$$\times [1 - w\overline{q}^{\sqrt{2n-1+y}}][1 - w^{-1}\overline{q}^{\sqrt{2n-1+y}}]\} \tag{14}$$

where

$$h(s) = (2\pi)^{-3/2} \int_{-\infty}^{\infty} \log(1 + s e^{-t^2})\, dt = \sum_{r=1}^{\infty} \frac{(-1)^{r-1} s^r}{\pi (2r)^{3/2}}. \tag{15}$$

Thus $h(-1) = -d$, $h(1) = c$.

Setting $x = 0$ and letting $y \to 1$, we obtain identity (4). (The $n = 0$ factor on the LHS, and the first two $n = 1$ factors on the RHS, vanish in this limit: we have to evaluate them to leading non-zero order and then take their ratio.)

As we have indicated above, this is sufficient to establish Kim's three identities (1)–(3). However, we also note that (14) contains all of (1)–(4) as special cases: (1) can be obtained by setting $x = -1$ and $y = 0$; (2) by setting $x = y = 0$; and (3) by setting $x = -1$ and $y = 1$.

Relation (14) plays a similar role to the conjugate modulus relation satisfied by the elliptic theta functions for arbitrary values of their argument. It is interesting to speculate whether the products therein, considered as functions of x or y, have any other properties resembling elliptic functions. For instance, are there algebraic relations corresponding to $\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$ or $k^2 + k'^2 = 1$?

Note added in proof. The author is indebted to George E Andrews for pointing out that equation (4) is stated (without reference or derivation) by G H Hardy and S Ramanujan in section 7.3 of their paper on asymptotic formulae in combinatorial analysis (1918 *Proc. London Math. Soc.* **17** 75–115). This is reprinted in volume 1 of Hardy G H 1966 *Collected Papers* (Oxford: Clarendon) p 335.

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